1.3: The Sentences of Predicate Logic

We now have all the pieces for saying exactly which expressions are going to count as sentences of predicate logic. First, all the sentences of sentence logic count as sentences of predicate logic. Second, we expand our stock of atomic sentences. I have already said that we will include among the atomic sentences predicates followed by the right number of names (one name for one place predicates, two names for two place predicates, and so on). We will do the same thing with variables and with variables mixed with names. So 'Bx' will count as an atomic sentence, as will 'Lxx', 'Lxy', and 'Lxa'. In general, any predicate followed by the right number of names and/or variables will count as an atomic sentence.

We get all the rest of the sentences of predicate logic by using connectives to build longer sentences from shorter sentences, starting from atomic sentences. We use all the connectives of sentence logic. And we add to these '(∀x)', '(∃y)', '(∃x)', '(∀y)', and other quantifiers, all of which count as new connectives. We use a quantifier to build a longer sentence from a shorter one in exactly the same way that we use the negation sign to build up sentences. Just put the quantifier in front of any expression which is already itself a sentence. We always understand the quantifier to apply to the shortest full sentence which follows the quantifier, as indicated by parentheses. Thus, if we start with 'Lxa', '(∀x)Lxa' counts as a sentence. We could have correctly written '(∀x)(Lxa)', though the parentheses around 'Lxa' are not needed in this case. To give another example, we can start with the atomic sentences 'Bx' and 'Lxe'. We build a compound by joining these with the conditional, '⊃', giving 'Bx ⊃ Lxe'. Finally, we apply '(∀x)' to this compound sentence. We want to be clear that '(∀x)' applies to the whole of 'Bx ⊃ Lxe', so we have to put parentheses around it before prefixing '(∀x)'. This gives '(∀x)(Bx ⊃ Lxe)'.

Here is a formal definition of sentences of predicate logic:

All sentence letters and predicates followed by the appropriate number of names and/or variables are sentences of predicate logic. (These are the atomic sentences.) If X is any sentence of predicate logic and u is any variable, then (∀u)X (a universally quantified sentence) and (∃u)X (an existentially quantified sentence) are both sentences...
of predicate logic. If \( X \) and \( Y \) are both sentences of predicate logic, then any expression formed from \( X \) and \( Y \) using the connectives of sentence logic are sentences of predicate logic. Finally, only these expressions are sentences of predicate logic.

Logicians often use the words \textit{Well Formed Formula} (Abbreviated \textit{wff}) for any expression which this definition classifies as a predicate logic sentence.

You may have noticed something a little strange about the definition. It tells us that an expression such as \((\forall x)Ba\) is a predicate logic sentence. If \( 'A' \) is a sentence letter, even \((\forall x)A\) is going to count as a sentence! But how should we understand \((\forall x)Ba\) and \((\forall x)A\)? Since the variable \( 'x' \) of the quantifier does not occur in the rest of the sentence, it is not clear what these sentences are supposed to mean.

To have a satisfying definition of predicate logic sentence, one might want to rule out expressions such as \((\forall x)Ba\) and \((\forall x)A\). But it will turn out that keeping these as official predicate logic sentences will do no harm, and ruling them out in the definition makes the definition messier. It is just not worth the effort to rule them out. In the next chapter we will give a more exact characterization of how to understand the quantifiers, and this characterization will tell us that "vacuous quantifiers," as in \((\forall x)Ba\) and \((\forall x)A\), have no effect at all. These sentences can be understood as the sentences 'Ba' and 'A', exactly as if the quantifiers were not there.

The definition also counts sentences such as ‘By’, ‘Lze’, and ‘Bx & Lxe’ as sentences, where ‘x’ and ‘z’ are variables not governed by a quantifier. Such sentences are called \textit{Open Sentences}. Open sentences can be a problem in logic in the same way that English sentences are a problem when they contain "open" pronouns. You fail to communicate if you say, 'He has a funny nose,' without saying or otherwise indicating who "he" is. Many logicians prefer not to count open sentences as real sentences at all. Where I use the expression 'open sentence', often logicians talk about 'open formulas' or 'propositional functions'. If you go on in your study of logic, you will quickly get used to these alternative expressions, but in an introductory course I prefer to keep the terminology as simple as possible.

Have you been wondering what the word 'syntax' means in the title of this chapter? The \textit{Syntax} of a language is the set of rules which tell you what counts as a sentence of the language. You now know what constitutes a sentence of predicate logic, and you have a rough and ready idea of how to understand such a sentence. Our next job will be to make the interpretation of these sentences precise. We call this giving the Semantics for predicate logic, which will be the subject of the next chapter. But, first, you should practice what you have learned about the syntax of predicate logic to make sure that your understanding is secure.

Exercise

1-3. Which of the following expressions are sentences of predicate logic?

a) Ca

b) Tab

C) aTb

d) Ca \rightarrow Tab
In the following exercises, use this transcription guide:

- a: Adam
- e: Eve
- c: Cid
- Bx: x is blond
- Cx: x is a cat
- Lxy: x loves y
- Txy: x is taller than y

Before you begin, I should point out something about transcribing between logic and pronouns in English. I used the analogy to English pronouns to help explain the idea of a variable. But that does not mean that you should always transcribe variables as pronouns or that you should always transcribe pronouns as variables. For example, you should transcribe 'If Eve is a cat, then she loves herself.' with the predicate logic sentence 'Ce ⊃ Lee'. Notice that 'she' and 'herself are both transcribed as 'e'. That is because in this case we have been told who she and herself are. We know that they are Eve, and so we use the name for Eve, namely, 'e' to transcribe these pronouns. How should we describe 'Ca ⊃ ~Ba'? We could transcribe this as 'If Adam is a cat then Adam is not blond.' But a nicer transcription is simply 'If Adam is a cat then he is not blond.'

Now do your best with the following transcriptions.

1-4. Transcribe the following predicate logic sentences into English:

a) ~Laa
b) Laa ⊃ ~Taa
c) ~(Bc v Lce)
d) Ca ≡ (Ba v Lae)
e) (∃x)Txc
f) (∀x)Lax & (∀x)Lcx
g) (∀x)(Lax & Lcx)
1-5. Transcribe the following English sentences into sentences of predicate logic:

a) Everyone loves Eve.

b) Everyone is loved by either Cid or Adam.

c) Either everyone is loved by Adam or everyone is loved by Cid.

d) Someone is taller than both Adam and Cid.

e) Someone is taller than Adam and someone is taller than Cid.

f) Eve loves all cats.

g) AU cats love Eve.

h) Eve loves some cats.

i) Eve loves no cats.

j) Anyone who loves Eve is not a cat.

k) No one who loves Eve is a cat.

l) Somebody who loves Adam loves Cid.

m) No one loves both Adam and Cid.

Chapter summary Exercise

Provide short explanations for each of the following. Check against the text to make sure that your explanations are correct, and keep your explanations in your notebook for reference and review.

a) Predicate Logic
b) Name

c) Predicate

d) One Place Predicate

e) Two Place Predicate

f) Relation

g) Variable

h) Universal Quantifier

i) Existential Quantifier

j) Universe, or Domain of Discourse

k) Govern

l) Bind

m) Open Sentence

n) Sentence of Predicate Logic

0) Well Formed Formula (wff)

p) syntax

q) Semantics